DIRECT AND INVERSE HEAT-CONDUCTION PROBLEMS FOR A SEMIINFINITE ROD FOR A PARTIAL OUTFLOW OF HEAT THROUGH THE SURFACE

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UDC 536.2.01

Analytical solutions of the direct and inverse problems of nonstationary heat conduction in a thin semiinfinite rod are given for the case of radiative heat fluxes at the lateral surfaces and a partial outflow of heat by convection and radiation through the end of the rod.

A long rod ($0 \leq x_1 < \infty$) of small cross-sectional area is heated by a flux of intensity $q(t_1)$ delivered to the face $x_1 = 0$. Part of this heat escapes through the lateral surfaces by convection and radiation into an external medium with constant temperature T_c. The nonstationary temperature distribution along the rod is found from the solution of the following boundary-value problem:

$$\frac{\partial T}{\partial t_1} = a\frac{\partial^2 T}{\partial x_1^2} - \frac{\alpha}{c\gamma h}[T(x_1, t_1) - T_c] - \frac{\varepsilon\sigma}{c\gamma h}[T^4(x_1, t_1) - T_c^4]; \tag{1}$$

$$\left(-\lambda\frac{\partial T}{\partial x_1}\right)_{x_1=0} = q(t_1), \ [T(x_1, t_1)]_{t_1=0} = \psi(x_1), \tag{2}$$

where $h = s/p$ is the ratio of the cross-sectional area to the perimeter of the rod (for a circular or square rod, $h = d/4$, where d is the diameter of the circle or side of the square). Introducing the temperature difference $U(x, t) = T(x, t) - T_c$, the problem reduces to

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 4\mathrm{Bi}\,U(x, t) - \mathrm{Sk}\,w(x, t); \tag{3}$$

$$\left(\frac{\partial U}{\partial x}\right)_{x=0} = -\frac{q(t)\,d}{\lambda}, \ [U(x, t)]_{t=0} = f(x) = \psi(x) - T_c, \tag{4}$$

where

$$w(x, t) = 4/T_c^3\,[T^4(x, t) - T_c^4].$$

Assuming that the thermal excitation functions $q(t)$, $f(x)$, $w(x, t)$ are given, the temperature field inside the rod is found by applying the Fourier cosine transform to the x dependence of (3) and (4). Then the problem reduces to:

$$U(x, t) = \frac{\exp(-4\mathrm{Bi}\,t)}{2\sqrt{\pi t}}\int_0^\infty f(\alpha)\left\{\exp\left[-\frac{(x+\alpha)^2}{4t}\right] + \exp\left[-\frac{(x-\alpha)^2}{4t}\right]\right\}d\alpha + \frac{d}{\lambda\sqrt{\pi}}\int_0^t \frac{\exp[-4\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}} \times$$
$$\times\exp\left[-\frac{x^2}{4(t-\tau)}\right]q(\tau)\,d\tau + \frac{\mathrm{Sk}}{2\sqrt{\pi}}\int_0^t\int_0^\infty \frac{\exp[-4\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}} \times \left\{\exp\left[-\frac{(x+\alpha)^2}{4(t-\tau)}\right] + \exp\left[-\frac{(x-\alpha)^2}{4(t-\tau)}\right]\right\} w(\alpha, \tau)\,d\tau d\alpha. \tag{5}$$

The solution of the direct problem of nonstationary heat conduction can be represented in terms of a functional operator H which depends on the functions q, f, and w:

$$T(x, t) = H[q(t), w(x, t), f(x), x, t].$$

Then if one of the functions in H is unknown and the other two are fixed, we can work out a method of solving the inverse problem of finding the unknown thermal excitation function, given the temperature field inside the bar.

Tadjik Polytechnic Institute, Dushanbe. Translated from Inzhenero-Fizicheskii Zhurnal, Vol. 47, No. 1, pp. 148-153, July, 1984. Original article submitted April 28, 1983.

 0022-0841/84/4701-0860$08.50

We consider a special case. Let $\psi(x) = T_0$ be the equilibrium initial distribution and $Sk = 0$. Then (5) reduces to

$$T(x, t) = T_c + (T_0 - T_c)\exp(-4\,\mathrm{Bi}\,t) + \frac{d}{\lambda\sqrt{\pi}}\int_0^t \frac{\exp[-4\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}}\exp\left[-\frac{x^2}{4(t-\tau)}\right] q(\tau)\,d\tau. \tag{6}$$

With the help of (6) the inverse problem of recovering the delivered heat flux (the sources) from measuring the temperature on the surface can be worked out. Let the measured temperature $T(x, t) - T_c$ on the surface $x = 0$ be approximated by a function $F(t)$. Putting $T_c = T_0$ in (6) and equating the right-hand side evaluated at $x = 0$ to $F(t)$, we obtain

$$F(t) = \frac{d}{\lambda\sqrt{\pi}}\int_0^t \frac{\exp[-4\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}} q(\tau)\,d\tau. \tag{7}$$

Introducing

$$k(t) = \exp(-4\,\mathrm{Bi}\,t)/\sqrt{t} \tag{8}$$

we obtain a Volterra integral equation of the first kind

$$F(t) = \frac{d}{\lambda\sqrt{\pi}}\int_0^t k(t-\tau)\,q(\tau)\,d\tau \tag{9}$$

with a kernel of the convolution type.

In the case of an insulated bar, Bi = 0 and (9) reduces to an Abel integral equation

$$F(t) = \frac{d}{\lambda\sqrt{\pi}}\int_0^t \frac{q(\tau)\,d\tau}{\sqrt{t-\tau}},$$

which has the known [1] solution

$$q(t) = \frac{\lambda}{d\sqrt{\pi}}\,\frac{d}{dt}\int_0^t \frac{F(\tau)}{\sqrt{t-\tau}}d\tau. \tag{10}$$

The solution of (9) for $\mathrm{Bi} \neq 0$ is found by the Laplace transform applied to an integral and reduces to the form

$$q(t) = \frac{\lambda}{d\sqrt{\pi}}\left\{\mathrm{Bi}\int_0^t \frac{F(\tau)\exp[-4\,\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}}\,d\tau + \int_0^t \frac{\exp[-4\,\mathrm{Bi}(t-\tau)]F'(\tau)}{\sqrt{t-\tau}}\,d\tau\right\}, \tag{11}$$

which in the limit $\mathrm{Bi} \to 0$ becomes

$$q(t) = \frac{\lambda}{d\sqrt{\pi}}\int_0^t \frac{F'(\tau)}{\sqrt{t-\tau}}\,d\tau,$$

and when $F(0) = 0$, this is equivalent to (10).

Let

$$F(t) = F_T(t) + \beta\theta(t), \tag{12}$$

where $F_T(t)$ corresponds to the exact temperature and $\beta\theta(t)$ represents the error in measurement and the approximation to the experimental curve (noise and errors in interpolation). Here β is a small parameter and $|\theta(t)| \leq 1$. Then after substitution of (12) into (11) we obtain

$$q(t) = q_T(t) + \Delta q(t), \tag{13}$$

where

$$\Delta q(t) = \frac{\lambda\beta}{d\sqrt{\pi}}\left\{\mathrm{Bi}\int_0^t \frac{\theta(\tau)\exp[-4\,\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}}\,d\tau + \int_0^t \frac{\theta'(\tau)\exp[-4\,\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}}\,d\tau\right\}; \tag{14}$$

and $q_T(t)$ is the exact heat flux found by substituting $F_T(t)$ into (11). When the condition $|\theta'(t)| \leqslant M$ is satisfied, we find the following estimate for the error in recovering the heat flux:

$$|\Delta q(t)| \leqslant \frac{\lambda\beta}{2d}\left[(\sqrt{\mathrm{Bi}} + M/\sqrt{\mathrm{Bi}})\,\mathrm{erf}(2\sqrt{\mathrm{Bi}\,t})\right],\ \beta > 0,$$

from which it follows that

$$\lim_{\beta\to 0}\Delta q(t)=0.$$

Thus for a semiinfinite rod, the determination of the heat flux from the measured temperature at a single point is a correctly posed problem and this is consistent with the general theory of inverse heat-conduction problems [2]. We point out that if the rod is not insulated so that Bi $\neq$ 0, the above calculation can be taken as a simplified method of solving the direct and inverse problems for a rod with a partial escape of heat through its surface by means of convection and radiation. Here it is necessary to use a somewhat overestimated value for Bi which will account for the fraction of heat given off by radiation.

We take $q(t) = -\varepsilon\sigma T^4(0, t)$. Then the temperature distribution inside the rod for the nonlinear boundary conditions of radiative heat exchange, using (5) with $f(x) = T_0 - T_c = \Delta T$ and Sk = 0, reduces to

$$T(x,\ t)-T_c=\Delta T\exp(-4\mathrm{Bi}\,t)-\frac{\varepsilon\sigma d}{\lambda\sqrt{\pi}}\int\limits_0^t\frac{\exp[-4\mathrm{Bi}\,(t-\tau)]\exp\left[-\frac{x^2}{4(t-\tau)}\right]}{\sqrt{t-\tau}}T^4(0,\ \tau)\,d\tau. \quad (15)$$

Putting x = 0, we obtain a nonlinear Volterra integral equation of the second kind for the function $\psi(t) = T(0, t)$:

$$\psi(t)=T_c+\Delta T\exp(-4\mathrm{Bi}\,t)-\frac{\varepsilon\sigma d}{\lambda\sqrt{\pi}}\int\limits_0^t k(t-\tau)\,\psi^4(\tau)\,d\tau, \quad (16)$$

where the kernel K(t) is determined as in (8).

If we put Bi = 0, then we have an insulated rod, and the problem is equivalent to the cooling of a one-dimensional semiinfinite body which radiates heat according to the Stefan–Boltzmann law from the surface x = 0. This problem was first considered and solved by Tikhonov [3]. Equation (16) for Bi = 0, and $T_c = T_0$ reduces to the form

$$\psi(t)=T_0-\frac{\varepsilon\sigma d}{\lambda\sqrt{\pi}}\int\limits_0^t\frac{\psi^4(\tau)}{\sqrt{t-\tau}}\,d\tau. \quad (17)$$

We introduce a substitution of variables

$$t=z/b^2,\ \tau=\xi/b^2,\ b=\mathrm{Sk}/\sqrt{\pi}=\varepsilon\sigma T_0^3 d/\lambda\sqrt{\pi}$$

and new functions

$$\Theta(z)=T(0,\ z/b^2)/T_0=\psi(z/b^2)/T_0,$$

then we obtain a nonlinear Volterra integral equation for $\Theta(z)$:

$$\Theta(z)=1-\int\limits_0^z\frac{\Theta^4(\xi)\,d\xi}{\sqrt{z-\xi}}, \quad (18)$$

whose solution exists and is unique [3]. By introducing a new function $\varphi(z)=\Theta^4(z)$ an approximate analytical solution of the integral equation (18) can be found by the method of successive approximations:

$$\varphi_n(z)=\left[1-\int\limits_0^z\frac{\varphi_{n-1}(\xi)}{\sqrt{z-\xi}}\,d\xi\right]^4. \quad (19)$$

In this method the iterations will give all $\varphi_n(z)$ less than $\varphi(z)$ when n is even and $\varphi_n(z)$ greater than $\varphi(z)$ for odd n, if we choose a first estimate $\varphi_0(z)$ which is known to be less than $\varphi(z)$. Hence the successive approximation scheme converges to the exact solution of (18) from both sides. After solving (18), we substitute the value $T^4(0,\ \tau)=T_0^4\Theta^4(b^2\tau)=T_0\varphi_n(b^2\tau)$ into (15) with Bi = 0, then the temperature distribution in a semiinfinite cooling body with outflow of heat from the surface x = 0 according to the Stefan–Boltzmann law is

$$T(x,\ t)=T_0-\frac{\varepsilon\sigma d}{\lambda\sqrt{\pi}}\int\limits_0^t\frac{\exp\left[-\frac{x^2}{4(t-\tau)}\right]T^4(0,\ \tau)}{\sqrt{t-\tau}}\,d\tau. \quad (20)$$

The cooling of a thin rod with initial temperature $T_0 \gg T_C$, when there is a convective and radiative outflow of heat from the lateral surfaces and when the heat outflow from the surface x = 0 is by radiation only reduces to the integral equation (16) with $Bi \neq 0$, for our simplified model. With no loss of generality, we put $T_C = 0$, then using variables z, ξ equation (16) reduces to

$$\Theta(z) = \exp(-4\mathrm{Bi}\, z/b^2) - \int_0^z k(z-\xi)\,\Theta^4(\xi)\,d\xi. \tag{21}$$

Introducing a new function $\Theta^4(\xi)\exp(4\mathrm{Bi}\,\xi/b^2) = \varphi(\xi)$, (21) becomes

$$\varphi(z) = \exp(-12\mathrm{Bi}\, z/b^2)\left[1 - \int_0^z \frac{\varphi(\xi)\,d\xi}{\sqrt{z-\xi}}\right]^4, \tag{22}$$

which gives the iteration formula

$$\varphi_n(z) = \exp(-12\mathrm{Bi}\, z/b^2)\left[1 - \int_0^z \frac{\varphi_{n-1}(\xi)}{\sqrt{z-\xi}}\,d\xi\right]^4. \tag{23}$$

Thus, the problem of recovering the heat flux (intensity of radiative heat outflow according to the Stefan–Boltzmann law from the surface x = 0 for a semiinfinite medium at initial temperature T_0, or for a rod with adiabatic conditions on the lateral surfaces (Bi = 0) or with uninsulated lateral surfaces ($Bi \neq 0$)) has been reduced to the solution of an integral equation without any additional information on the temperature variation at a single point. Problems of this type can be called pseudoinverse heat-conduction problems [2].

Equation (5) can be used to calculate the temperature when there is a concentrated exposure of the surface x = 0 to radiation according to Bouguer's law [4]:

$$q(t) = \varepsilon(t)(1-\rho), \tag{24}$$

where $\varepsilon(t)$ is the irradiance (a given function of time) and ρ is the coefficient of reflection; $A_q = 1 - \rho$ is the coefficient of absorption for the irradiated surface. We give the solutions when $\varepsilon(t)$ is a constant and an exponential. We substitute $q(t) = q = \varepsilon(t)(1 - \rho) = \mathrm{const}$ in (5). Then for f(x) = 0, ($T_0 = T_C$), and Sk = 0, the temperature distribution over the length of the rod for a constant heat flux has the form:

$$T(x, t) = T_0 + \frac{qd}{\lambda\sqrt{\pi}}\int_0^t \frac{\exp[-4\mathrm{Bi}(t-\tau)]\exp\left[-\frac{x^2}{4(t-\tau)}\right]}{\sqrt{t-\tau}}\,d\tau. \tag{25}$$

The integral on the right-hand side can be expressed in terms of the error function [5] and the solution (25) takes the form

$$T(x, t) = T_0 + qd/4\lambda\sqrt{\mathrm{Bi}}\,\{\exp(-2\sqrt{\mathrm{Bi}}\,x)\,\mathrm{erfc}(x/2\sqrt{t} - 2\sqrt{\mathrm{Bi}\,t}) - \exp(2\sqrt{\mathrm{Bi}}\,x)\,\mathrm{erfc}(x/2\sqrt{t} + 2\sqrt{\mathrm{Bi}\,t})\}. \tag{26}$$

In the limit Bi → 0, the indeterminate form 0/0 is evaluated according to L'Hospital's rule and the temperature distribution inside a rod with an adiabatic surface has the form

$$T(x, t) = T_0 + qd/\lambda\left[\mathrm{erfc}(x/2\sqrt{t}) - \frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{\pi}\,x/2\sqrt{t}}\right], \tag{27}$$

which agrees with the result of [6].

For an exponential heat flux

$$q(t) = \varepsilon(t)(1-\rho) = q(1-\rho)[1 - \exp(-\nu t)]$$

we find

$$T(x, t) = T_0 + \Phi(x, t) - \frac{2q(1-\rho)\exp(-\nu t)}{\lambda\sqrt{\pi}(4\mathrm{Bi}-\nu)}\{\exp(-\sqrt{4\mathrm{Bi}-\nu}\,x)\times$$
$$\times\,\mathrm{erfc}(x/2\sqrt{t} - \sqrt{4\mathrm{Bi}-\nu t}) - \exp(\sqrt{4\mathrm{Bi}-\nu}\,x)\,\mathrm{erfc}(x/2\sqrt{t} + \sqrt{4\mathrm{Bi}-\nu t})\}. \tag{28}$$

The general problem (1), (2) with given heat fluxes q(t) can, with the help of (5), be reduced to the solution of a nonlinear two-dimensional Volterra–Fredholm integral equation of the second kind for the temperature

$$T(x,\ t)-T_c=\Phi(x,\ t)-\frac{\mathrm{Sk}}{2\sqrt{\pi}\,T_0^3}\int_0^t\int_0^\infty k(x,\ t,\ \alpha,\ \tau)[T^4(\alpha,\tau)-T_c^4]\,d\tau d\alpha, \tag{29}$$

where

$$\Phi(x,\ t)=\frac{d}{\lambda\sqrt{\pi}}\int_0^t\frac{\exp[-4\mathrm{Bi}(t-\tau)]\exp\left[-\frac{x^2}{4(t-\tau)}\right]}{\sqrt{t-\tau}}\,q(\tau)\,d\tau,$$

$$K(x,\ t,\ \alpha,\ \tau)=\frac{\exp[-4\mathrm{Bi}(t-\tau)]}{\sqrt{t-\tau}}\left\{\exp\left[-\frac{(x+\alpha)^2}{4(t-\tau)}\right]+\exp\left[-\frac{(x-\alpha)^2}{4(t-\tau)}\right]\right\}.$$

Equation (29) is written for the case $f(x) = 0$ ($T_0 = T_c$).

Finally, some of these solutions have been used to study the temperature distributions in rods heated by intense radiative fluxes from solar concentrators. An application of these studies is in the technology of crystal growing using solar energy.

NOTATION

a, thermal diffusivity; x_1, coordinate along the length of the rod; t_1, time; $t = at_1/d^2$, dimensionless time (Fourier number); $x = x_1/d$, relative coordinate; T_0, initial temperature; σ, Boltzmann constant; $\mathrm{Sk} = \varepsilon\sigma T_c^3 d/\lambda$, Stark number; $\mathrm{Bi} = \alpha d/\lambda$, reduced Biot number; ε, emissivity.

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